Appendix

It is one of Koopman's major achievements to have given precise, operational meaning to the notion (already familiar in common usage) of judging a probability to lie within an interval of values. He does this by introducing a concept which is to lead, in special cases, to the definite assignment of numbers to probabilities, and in general to the assignment of upper and lower numerical bounds: the concept of an "n-scale."

Definition: An n-scale is defined as any set of n propositions $(u_1, \ldots\ldots, u_n)$ for which the following is posited:

1) $u = u_1 \vee \ldots\ldots \vee u_n \neq o.$

2) $u_i u_j = o \; (i \neq j)$ for all $i, j = 1, \ldots\ldots, n.$

3) $u_i/u \approx u_j/u$ for all $i, j = 1, \ldots\ldots, n.$

In words, u is a non-null set of propositions, every pair of which are mutually exclusive and are judged by a given individual, on the presumption u, to be equally probable. Since this involves a subjective judgement of equal probability, a set of n propositions that constitutes an n-scale for one person need not qualify as such for another; and it is an empirical proposition to assert that a given set of propositions satisfies these conditions for a given person. No assertion is made by Koopman that any given set of propositions, or any set defined purely by its "objective" (impersonal) properties, "should" or will meet these conditions for all individuals or for any given individual; there is, in other words, no introduction of a Principle of Indifference, Insufficient Reason, Cogent Reason, etc., justifying either the prediction or the prescription of the assignment of equal probabilities to a given set of propositions

(eventualities). He merely proposes to consider the implications of the hypothesis that some such sets do exist for a given individual.

It is important to spell out the role of this assumption in Koopman's analysis. Prior to introducing this concept, he develops a theory of probability (proving, for example, thirteen theorems on comparisons of probability from his axioms) based solely upon the notion of comparisons, or judged inequalities, between the probabilities of different eventualities. This theory places powerful--yet intuitively compelling--restraints on the structure of a reasonable body of beliefs, while remaining a wholly non-numerical theory. None of Koopman's fundamental axioms even involves the notion that two probabilities are judged to be equal (only the relation $\leq$ signifying "equally or less probable than" is employed). Koopman demonstrates that it is not necessary to the development of a powerful theory along these lines to suppose that a given individual ever makes such a judgment of equality. (The nature of the propositions on inequalities of belief that are presumed or that may be deduced from such a theory is indicated by the two axioms, of transitivity and antisymmetry, cited in the earlier quotation. Thus the presumed existence of even one n-scale--or even, one single judgment of equal probability between two non-certain and non-null propositions--is neither a requirement of "reasonableness," nor a prediction concerning typical bodies of belief, nor a proposition put forward as "intuitively evident" in the spirit of the axioms. Rather, the existence of an n-scale is "assumed" for the purpose of exploring its implications in the light of the axioms and theorems.

To this end, Koopman actually uses a stronger assumption: that if n is any positive integer, at least one n-scale may be regarded as existing

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(for a given individual). He never offers, as an aid to intuition, a speciman set of eventualities as promising candidates for n-scale status, but I. J. Good has suggested two familiar embodiments of this notion:

> "Numerical probabilities can be introduced by imagining perfect packs of cards perfectly shuffled, or infinite sequences of trials under essentially similar conditions. Both methods are idealizations, and there is very little to choose between them. It is a matter of taste: that is why there is so much argument about it."[1]

Thus, the eventuality u_i/u can be interpreted in a particular setting: "Given that one of these n perfectly shuffled cards numbered 1 to n is to be drawn, the card numbered i will be the one selected." Or alternatively: "Given that a fair coin will be tossed under similar conditions $\log_2 n$ times so that these n sequences of Heads, Tails are possible, the ith sequence (e.g., Heads, Heads, Tails.....Heads) will occur." The existence of an n-scale for a given individual would be demonstrated if a pack of n cards or a coin susceptible of log n tosses could be produced, or imagined by him, such that he was willing to assert that he judged all n eventualities, u_i/u, u_j/u (interpreted as above) to be equally probable. The assumption that such packs or coins can be produced or imagined for all n (n indefinitely large) is, as Good says, obviously an idealization of reality, but also obviously, one that is familiar and seems permissible for purposes of conceptual discussion.

Koopman then states: Theorem 14. If $(u_1, \ldots u_n)$ is an n-scale and $(v_1, \ldots v_m)$ an m-scale, and if ρ and σ are integers with $0 \leq \sigma \leq n$, $0 \leq \rho \leq m$, then $u_{i1} \vee \ldots \vee u_{i\sigma}/u <, \approx, > v_{j1} \vee \ldots \vee v_{j\rho}/v$ according as $\sigma/n <, =, > \rho/m$. Here it is understood that $i_1, \ldots, i_\sigma$ are each

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[1] Good, "Rational Decision," p. 110.

between 1 and n, and that no two are equal; similarly for $j_1 \ldots j_\rho$ and m; the symbol $<$ between eventualities stands for the combined assertion of $a/h \prec b/k$ and denial of $a/h \succ b/k$, i.e., the assertion that a/h is strictly less probable than b/k; similarly for the symbol $>$. This theorem follows from Koopman's axioms on comparisons and his definition of an n-scale, but its meaning is fairly evident in terms of the suggested interpretations of n-scales above. In particular, if n=m (whether we are dealing with two different scales with the same number n of members, or a given set of n propositions) it asserts that the individual can always judge the comparative probability of (a) the combined assertion of any subset of ρ propositions from the set of n propositions, and (b) the combined assertion of any subset of σ propositions, and that the relative probability will depend only upon the numerical magnitudes of ρ, σ, not upon the particular choice of propositions asserted. For example, if we imagine 100 cards perfectly shuffled and numbered 1,.....100, any assertion "Given that one of these cards will be drawn, one of the three cards i, j, k will be drawn" will be judged equally probable to any similar assertion for any specification of the three numbers i, j, k (e.g., 1, 2, 3; 98, 99 100; 7, 19, 54; etc.). And any one of these assertions will be judged strictly less likely than an assertion that one of five cards will be drawn, for any specification of the five numbers (all of this on the assumption that assertions concerning drawings from this pack of cards fulfill the conditions of an n-scale for this individual, who obeys the Koopman axioms on comparisons of probability).

Koopman now approaches concepts and results of extreme usefulness and breadth of application. Let us consider a given eventuality, a/h, an

an actual test, if I were asked successively to compare the probability of Dick's election to the draw of one out of 5, 6..., t cards, there would be a certain arbitrariness to the exact value of *t* at which I was no longer able to assert a definite judgment that Dick's election was equally or more probable, since I do not know (and, frankly, do not wish to know) my own mind well enough to make judgments of that precision. But without going into all the subtleties of actual experimental procedure suitable to such problems, let us assume that in a given case the "zone of arbitrariness" (within which I fluctuate on successive tests, or have to ponder especially hard, or some other operational criterion) is fairly narrowly limited, say to (10 plus or minus 2). In fact, to abstract from this problem for the moment, let us follow Koopman in assuming that a grestest lower bound t(n) can be determined, e.g., t(n) = 10. This means merely that where the possibility of drawing one of 11 or more specified cards out of 100 is concerned, I am *no longer able to say* definitely that I consider Dick's election equally or *more* probable.

Now we can define a value T(n) in precisely analogous fashion as the minimum number T for which it is true that: $a/h < u_1 \vee \ldots\ldots \vee u_T/u$. We must keep in mind that where the relation $<$ is concerned, it implies a definite judgment of "equally or less probable than" and a reasonable person is not committed to producing such a judgment with respect to every pair of propositions; in a given case he may assert *neither* $a/h < b/k$ nor $b/k < a/h$, so that his inability or unwillingness to assert the one may not be taken to imply that he is able or willing to assert the other. We assume that a person will be able to make the above assertion for *some* number T, if only for the value T=n. Moreover, we assume that there is a *minimum*

number $T(n)$ for which the person will be willing to assert the comparison; again, in practice this will be determined only roughly and somewhat arbitrarily, but we shall assume the assignment made. It will not be less than $t(n)$, considering the way in which $t(n)$ was determined, so we have $0 \leqq t(n) \leqq T(n) \leqq n$.

In terms of our example, let us suppose that my willingness to assert definitely that Dick's election is equally or <u>less</u> probable than the draw of one out of T cards <u>ceases</u> when T is reduced, roughly, below 30 cards.

Theorem 15: <u>The following limits always exist</u>:

$$p_*(a,h) = \lim_{n\to\infty} \frac{t(n)}{n}\ ; \quad p^*(a,h) = \lim_{n\to\infty} \frac{T(n)}{n}\ ;$$

<u>and they satisfy the relation</u>

$$0 \leqq p_*(a,h) \leqq p^*(a,h) \leqq 1.$$

Thus, if we imagine a pack of 1000 cards, $t(n)$ might be 105, or to keep the arithmetic simple, let us say 100; if we take 100/1000 as an approximation to the limiting value of $t(n)$, we have: $p_*(a, h) = 1/10$. **This** may be read: My <u>lower probability</u> that Dick will be elected, given that Dick or Jack will be elected, is 1/10. The operational meaning of this statement is contained in the specification of the particular comparisons that led up to it, along with the axioms and theorems on which inferences were based. If propositions about drawings from a given pack of 100 shuffled cards constitute a 100-scale for me, it amounts to the proposition, with appropriate pragmatic qualifications added: I will <u>stop</u> asserting definitely that I think Dick's election is equally or <u>more</u> probable than drawing one of t specified cards, for increasing <u>t</u>, when t is greater than (about) 10. And if we take $p^*(a, h)$ to be 3/10, this

translates roughly: I will <u>no longer</u> be able to assert definitely that Dick's election is equally or <u>less</u> probable than a drawing of one out of T cards, for decreasing T, when T is less than (roughly) 30.

I have supplied the "sloppiness" in the above exposition. Koopman's own statements, to which the reader is referred, are rigorous and precise, but limited to the implications of his abstract, idealized assumptions. He makes none of the concessions to the demands of application that would be required in even the most scrupulous empirical interpretation of these concepts. The reason I have added some of these qualifications, even in this initial, hypothetical attempt at interpretation, is to make clear the distinction between, on the one hand, the inevitable "fuzziness" surrounding the values of $p_*(a,h)$ and $p^*(a,h)$ and, on the other, the potential existence of a definite, distinctly large <u>interval between</u> these two. The precise length of this interval, $p^*(a,h) - p_*(a,h)$, will itself be "fuzzy," vague, arbitrary, indeterminate, to the degree that this is true of its two end-points. But this leaves open the possibility that the <u>smallest</u> assignment of length to this interval consistent with the evidence accepted in a given empirical application will be <u>significantly greater than 0</u>; even though, within the same standards of empirical interpretation, the person making the judgments (or refraining from them) strictly obeys all of the axioms and theorems laid down by Koopman for coherent, "reasonable" comparisons of probability and inferences therefrom.

The axioms supplied by Koopman are described by him as "<u>the</u> intuitively evident laws of consistency governing all comparisons in probability"[1]; it is implied that no additional axioms justifying stronger inferences

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[1] Koopman, <u>op</u>. <u>cit</u>., p. 275, italics added.

qualify, in his opinion, for assertion with the same intuitive force. Moreover, these axioms, in his opinion, "form a <u>sufficient</u> basis for what we envision as the legitimate role of a theory of intuitive probability", his conception of this role corresponding to the one we have assumed. But the <u>strongest</u> inferences that these axioms and theorems, together with the assumption of the existence of n-scales, permit us to draw in general about the relationship of upper and lower probabilities for (a , h) are the <u>inequalities</u> given in Theorem 15. <u>They do not imply, in general, that</u> $p_*(a,h) = p^*(a,h)$; this equality is not a requirement, in Koopman's approach, of a "reasonable" or "consistent" body of beliefs, i.e., it does not follow from what he considers to be the intuitively appropriate laws of consistency for comparison of intuitive probability.

Two more theorems are important for our understanding of this approach.

measurement and judgment are ignored: when there is assumed to be a sharp, subjective boundary between "ability to compare" and "inability to compare."

To say that there is a positive interval between $p_*(a, h)$ and $p^*(a, h)$--ignoring, for the moment, questions of imprecision in the determination of these boundary values, or else postulating that the length surpasses the possible combined effect of such imprecision--is to say that for any given *n*-scale, there are numbers *s* between $t(n)$ and $T(n)$ such that the individual can or will judge *neither* that $a/h < u_1 \vee \ldots\ldots \vee u_{s/u}$, *nor* $a/h > u_1 \vee \ldots\ldots \vee u_{s/u}$. In other words, there are pairs of (complex) propositions which the individual is *unable to compare* with respect to relative probability; he is unable to judge either that one is more probable than the other, or that they are equally probable. Hence the description of $<$ as a *partial* ordering.

This is the implication of the numbers hypothesized in the example mentioned earlier. To say that $t(100) = 10$ and $T(100) = 30$ for the (a, h) considered, means that if I am called upon to compare the probability of Dick's election with that of your drawing one of, say, 20 cards out of 100 in the pack, I will *not* indicate that I regard Dick's success as equally or more likely; *nor* will I indicate that I regard it as equally or more likely. (Doing neither, of course, is not the same as doing both: which would mean I considered these propositions equally probable.) I use the word "indicate" here instead of "assert" as formerly, to make clear

Moreover, for such eventualities, the usual axioms of a classical probability measure will hold for $p(a, h)$; e.g.,

Theorem 24. If $a_1 a_2 h = 0$ and a_1/h and a_2/h are both appraisable, then $a_1 \vee a_2/h$ will be appraisable, and

$$p\ (a_1 \vee a_2,\ h) = p\ (a_1,\ h) + p\ (a_2,\ h).$$

In particular, as one might expect, eventualities in which the presumption is an n-scale and the contingencies are unions of disjoint elements are appraisable:

Theorem 20. If $(a_1, \ldots\ldots, a_r)$ is a v-scale $(V = 1, 2\ldots\ldots)$ and if b = sum of T distinct elements a_i:

$$b = a_{i_1} \vee \ldots\ldots \vee a_{i_T} \quad (b = 0 \text{ for } T = o)$$

then b/a is appraisable, and $p\ (b,\ a) = T/r$.

Koopman makes the important remark that if all eventualities were completely ordered by $<$, (assuming the existence of n-scales for all n) every eventuality would be appraisable.

The consequences of the presumed non-appraisability of the eventualities in our example can now be seen more clearly. Assuming the existence of a coin I consider fair, then a set of judgments that would be compatible with my earlier answers would be: "Compared to the election of Dick, on the presumption that Dick or Jack will be elected, the occurrence of Heads on a single toss is more probable; the occurrence of, say, Heads four times in four tosses (or any other specified sequence of four outcomes) is less probable; the occurrence of two Heads in two tosses or three Heads in three tosses cannot be definitely compared in probability" whatever the verbal or behavioral correlation of inability to judge). Consistently, in terms of Theorem 16, I should judge that the occurrence of Tails (not

Heads) on a single toss is <u>less</u> probable than the election of Jack (not-(election of Dick)), but that it is <u>more</u> probable that Heads will <u>not</u> occur four times in four tosses than that Jack will win; and corresponding pairs should be non-comparable.